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THE POYNTING-ROBERTSON EFFECT AND SECULAR CHANGES OF ORBITAL ELEMENTS

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Abstract. The Poynting-Robertson (P-R) effect has been applied to meteoroids for several decades. It is well known that the P-R effect produces only changes in the orbital plane of the particle. The differential equations governing the secular changes in both eccentricity and semi-major axis are known since the time of Robertson, provided the initial orbits are not near circular. A disadvantage of this type of osculating orbital elements is that it cannot be applied to motion of arbitrarily shaped particles. Relevant type of osculating orbital elements is discussed. It's application to the P-R effect is presented and simple analytical formulae for secular changes in both eccentricity and semi-major axis are derived.

1. Introduction

The relativistic equation of motion for a perfectly absorbing spherical dust particle under the action of electromagnetic radiation was obtained by Robertson (1937). Robertson derived expressions for the secular changes in the orbital elements. The effect essentially reduces both semi-major axis and eccentricity and is popularly known as the Poynting-Robertson (P-R) effect. The results of Robertson were verified by Wyatt and Whipple (1950) and applied to the evolution of meteoroid streams. One particularly important aspect, first pointed out by Harwit (1963), is that a meteoroid escapes from the Solar

System when the orbital energy becomes positive and that this can happen when the energy due to the radial component of the radiation force is included, without it being necessary for this force to exceed the gravitational attraction.

A relativistic generalization corresponding to the P-R effect was discussed by Klačka (1992a). A more simple derivation is given in Klačka (1993a). There have been many misunderstanding, and even errors, in the treatment of radiation force effects in the litterature and some of these are discussed in Klačka (1993b, 2000b). We will not dwell further on most of these here. A consistent set of equations governing, to the first order in $\boldsymbol{v}/c$, the secular changes in the orbital elements were given by Klačka (1992b) (here, $\boldsymbol{v}$ is velocity of the particle, c is the speed of light), where the appropriate initial conditions were also discussed. This work, and most of the preceeding work assumes that the orbits are not near circular. When the orbits are near circular, different considerations must apply, for example a circular orbit can not reduce in radius without increasing in eccentricity, contrary to the normal behaviour under P-R effect. The near-circular situation has been discussed by Klačka and Kaufmannová (1992, 1993); the results analytically confirmed Breiter and Jackson (1998) and their nonphysical conclusions are discussed in Klačka (2000a, 2001a). We will not consider this near-circular case in detail, but discussion will contain a short comment.

The Poynting-Robertson effect also causes an advancement of perihelion, although the the effect is second order in $\boldsymbol{v}/c$ (that is, the secular change is proportional to c^{-2}). Correct result can be found in Balek and Klačka (2002).

The aim of this paper is to discuss relevant form(s) of osculating orbital elements and their secular changes. The motivation comes from the fact that the P-R effect represents only a very special case of interaction between a particle and incident electromagnetic radiation (Klačka 2000c, 2000d, 2000e, 2001b, 2002a, 2002c, Klačka and Kocifaj 2001). Type of secular changes of osculating orbital elements derived by Robertson (1937) cannot be compared with secular changes of these elements for real particles: secular changes of osculating orbital elements based on central gravity force alone, are relevant physical quantities (Klačka 2002b). The correct equations governing the secular evolution of semi-major axis and eccentricity are derived for the P-R effect.

2. P-R effect: Equation of motion

Consider a spherical particle with a geometric cross section A' in a flow of incident radiation with flux density S in a direction denoted by a unit vector $\boldsymbol{S}$. Let Q'_{pr} denote the efficiency factor for radiation. As is usual within a relativistic context, we use $\gamma = 1/\sqrt{1 - \boldsymbol{v}^2/c^2}$. It is convenient to define $w = \gamma(1 - \boldsymbol{v} \cdot \boldsymbol{S}/c)$.

The relativistically covariant equation of motion for the P-R effect may be expressed as (Klačka 1992a, 2000c, 2002a, 2002c)

$$\frac{d\, p^{\mu}}{d\, \tau} = \frac{w^2\ S\ A'}{c^2}\ Q'_{pr}\ \left(c\, b^{\mu}\ -\ u^{\mu}\right)\ , \tag{1}$$

where p^{μ} is four-vector of the particle of mass m, $p^{\mu} = m\ u^{\mu}$, four-vector of the world-velocity of the particle is $u^{\mu} = (\gamma\ c, \gamma\ \boldsymbol{v})$ and $b^{\mu} = (1/w)\ (1, \boldsymbol{S})$.

Relativistically covariant form of equation of motion is important in deriving an expression for the secular change of an advancement of perihelion, which can be found in Balek and Klačka (2002). Moreover, higher orders in v/c play an important role when making calculations for near circular orbits (compare Breiter and Jackson 1998, Klačka 2000a, 2001a). This will not be considered here, except section 3.1. Thus, reduction of Eq. (1) to the first order in $\boldsymbol{v}/c$ will be used, if it is not explicitly stressed.

Applying Eq. (1) to the situation where a particle is moving in gravitational and electromagnetic fields of a star (usually the Sun) with mass M gives

$$\frac{d\boldsymbol{v}}{dt} = -\frac{\mu}{r^2}\boldsymbol{S} + \beta\frac{\mu}{r^2}\left\{\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{S}}{c}\right)\boldsymbol{S} - \frac{\boldsymbol{v}}{c}\right\}\ , \tag{2}$$

where $\mu = GM$, G being the gravitational constant, β is the ratio of radiation pressure force to gravity force, that is $\beta = Q'_{pr}\ A'\ r^2\ S\ /\ (c\ \mu\ m)$.

3. Secular changes of orbital elements – radiation pressure as a part of central acceleration

We have to use $-\ \mu\ (1\ -\ \beta)\ \boldsymbol{S}\ /\ r^2$ as a central acceleration determining osculating orbital elements if we want to take a time average (T is time interval between passages through two following pericenters) in an analytical way

$$\begin{aligned}
\langle g \rangle\ &\equiv\ \frac{1}{T}\int_0^T g(t)dt = \frac{\sqrt{\mu\ (1-\beta)}}{a_{\beta}^{3/2}}\ \frac{1}{2\pi}\int_0^{2\pi} g(f_{\beta})\left(\frac{df_{\beta}}{dt}\right)^{-1} df_{\beta} \\
&= \frac{\sqrt{\mu\ (1-\beta)}}{a_{\beta}^{3/2}}\ \frac{1}{2\pi}\int_0^{2\pi} g(f_{\beta})\ \frac{r^2}{\sqrt{\mu\ (1-\beta)\ p_{\beta}}}\ df_{\beta} \\
&= \frac{1}{a_{\beta}^2\ \sqrt{1 - e_{\beta}^2}}\ \frac{1}{2\pi}\ \int_0^{2\pi}\ g(f_{\beta})\ r^2\ df_{\beta}\ ,
\end{aligned} \tag{3}$$

assuming non-pseudo-circular orbits and the fact that orbital elements exhibit only small changes during the time interval T; a_{β} is semi-major axis, e_{β} is eccentricity, f_{β} is true anomaly, $p_{\beta} = a_{\beta}(1 - e_{\beta}^2)$; the second and the third Kepler's laws were used: $r^2\ df_{\beta}/dt =$

$\sqrt{\mu(1-\beta)p_\beta}$ – conservation of angular momentum, $a_\beta^3/T^2 = \mu(1-\beta)/(4\pi^2)$. (For more details see Klačka, 1992b.)

Rewriting Eq. (2) into the form

$$\frac{d\boldsymbol{v}}{dt} = - \frac{\mu \ (1-\beta)}{r^2} \boldsymbol{S} - \beta \frac{\mu}{r^2} \left\{ \left(\frac{\boldsymbol{v} \cdot \boldsymbol{S}}{c} \right) \boldsymbol{S} + \frac{\boldsymbol{v}}{c} \right\} \ , \tag{4}$$

we can immediately write for components of perturbation acceleration to Keplerian motion:

$$F_{\beta \ R} = - \ 2 \ \beta \ \frac{\mu}{r^2} \ \frac{v_R}{c} \ , \quad F_{\beta \ T} = - \ \beta \ \frac{\mu}{r^2} \ \frac{v_T}{c} \ , \quad F_{\beta \ N} = 0 \ , \tag{5}$$

where $F_{\beta \ R}$, $F_{\beta \ T}$ and $F_{\beta \ N}$ are radial, transversal and normal components of perturbation acceleration, and two-body problem yields

$$\begin{aligned} v_R \ &= \ \sqrt{\frac{\mu \ (1-\beta)}{p_\beta}} \ e_\beta \sin f_\beta \ , \\ v_T \ &= \ \sqrt{\frac{\mu \ (1-\beta)}{p_\beta}} \ \left(1 + e_\beta \cos f_\beta \right) \ . \end{aligned} \tag{6}$$

The important fact that pertubation acceleration is proportional to v/c ($\ll 1$) ensures the above metioned small changes of orbital elements during the time interval T.

Perturbation equations of celestial mechanics yield for osculating orbital elements (a_β – semi-major axis; e_β – eccentricity; i_β – inclination (of the orbital plane to the reference frame); Ω_β – longitude of the ascending node; ω_β – longitude of pericenter; Θ_β – $\Theta_\beta = \omega_\beta + f_\beta$ is the position angle of the particle on the orbit, when measured from the ascending node in the direction of the particle's motion):

$$\begin{aligned}
\frac{da_\beta}{dt} \ &= \ \frac{2 \ a_\beta}{1 \ - \ e_\beta^2} \ \sqrt{\frac{p_\beta}{\mu \, (1 \ - \ \beta)}} \ \left\{ F_{\beta \ R} \, e_\beta \ \sin f_\beta + F_{\beta \ T} \, (1 \ + \ e_\beta \ \cos f_\beta) \right\} \ , \\
\frac{de_\beta}{dt} \ &= \ \sqrt{\frac{p_\beta}{\mu \, (1 \ - \ \beta)}} \ \left\{ F_{\beta \ R} \ \sin f_\beta + F_{\beta \ T} \left[\cos f_\beta \ + \ \frac{e_\beta + \cos f_\beta}{1 + e_\beta \cos f_\beta} \right] \right\} \ , \\
\frac{di_\beta}{dt} \ &= \ \frac{r}{\sqrt{\mu \, (1 \ - \ \beta) \, p_\beta}} \ F_{\beta \ N} \ \cos \Theta_\beta \ , \\
\frac{d\Omega_\beta}{dt} \ &= \ \frac{r}{\sqrt{\mu \, (1 \ - \ \beta) \, p_\beta}} \ F_{\beta \ N} \ \frac{\sin \Theta_\beta}{\sin i_\beta} \ , \\
\frac{d\omega_\beta}{dt} \ &= \ - \ \frac{1}{e_\beta} \ \sqrt{\frac{p_\beta}{\mu \, (1 \ - \ \beta)}} \ \left\{ F_{\beta \ R} \cos f_\beta - F_{\beta \ T} \frac{2 + e_\beta \cos f_\beta}{1 + e_\beta \cos f_\beta} \sin f_\beta \right\} \ - \\
& \qquad \frac{r}{\sqrt{\mu \, (1 \ - \ \beta) \, p_\beta}} \ F_{\beta \ N} \ \frac{\sin \Theta_\beta}{\sin i_\beta} \ \cos i_\beta \ , \\
\frac{d\Theta_\beta}{dt} \ &= \ \frac{\sqrt{\mu \, (1 \ - \ \beta) \, p_\beta}}{r^2} \ - \ \frac{r}{\sqrt{\mu \, (1 \ - \ \beta) \, p_\beta}} \ F_{\beta \ N} \ \frac{\sin \Theta_\beta}{\sin i_\beta} \ \cos i_\beta \ ,
\end{aligned} \tag{7}$$

where $r = p_\beta / (1 + e_\beta \cos f_\beta)$.

Inserting Eqs. (5) – (6) into Eq. (7), one easily obtains

$$
\begin{aligned}
\frac{da_\beta}{dt} &= -\,\beta \frac{\mu}{r^2}\frac{2a_\beta}{c}\frac{1+e_\beta^2+2e_\beta\cos f_\beta+e_\beta^2\sin^2 f_\beta}{1\,-\,e_\beta^2}\ ,\\
\frac{de_\beta}{dt} &= -\,\beta \frac{\mu}{r^2}\frac{1}{c}\left(2e_\beta+e_\beta\sin^2 f_\beta+2\cos f_\beta\right)\ ,\\
\frac{di_\beta}{dt} &= 0\ ,\\
\frac{d\Omega_\beta}{dt} &= 0\ ,\\
\frac{d\omega_\beta}{dt} &= \beta \frac{\mu}{r^2}\frac{1}{c}\frac{1}{e_\beta}\ \left(2-e_\beta\cos f_\beta\right)\sin f_\beta\ ,\\
\frac{d\Theta_\beta}{dt} &= \frac{\sqrt{\mu\left(1-\beta\right)p_\beta}}{r^2}\ .
\end{aligned}
\tag{8}
$$

It is worth mentioning that $da_\beta/dt < 0$ for any time t (we again stress that near circular orbits are not considered).

The set of differential equations Eqs. (8) has to be complemented with initial conditions. If the subscript 0 denotes orbital elements of the parent body (e. g., comet), then even if a particle is ejected with zero relative velocity, it will move on a different orbit due to the central force on it being reduced because of the radiation. (For more general case, with non-zero ejection, see Gajdošík and Klačka 1999.) The elements of the "new" orbit are given by

$$
a_{\beta\ in} = a_0\left(1-\beta\right)\left(1-2\beta\frac{1+e_0\cos f_0}{1\,-\,e_0^2}\right)^{-1}\ , \tag{9}
$$

$$
e_{\beta\ in} = \sqrt{1-\frac{1-e_0^2-2\beta\left(1+e_0\cos f_0\right)}{\left(1-\beta\right)^2}}\ , \tag{10}
$$

where $f_0 \equiv \Theta_0 - \omega_0$, $\omega_{\beta\ in}$ has to be obtained from

$$
\begin{aligned}
e_{\beta\ in}\ \cos\left(\Theta_0-\omega_{\beta\ in}\right) &= \frac{\beta+e_0\cos f_0}{1-\beta}\ ,\\
e_{\beta\ in}\ \sin\left(\Theta_0-\omega_{\beta\ in}\right) &= \frac{e_0\sin f_0}{1-\beta}\ ,
\end{aligned}
\tag{11}
$$

$$
\Omega_{\beta\ in} = \Omega_0\ ,\quad i_{\beta\ in} = i_0\ ,\quad \Theta_{\beta\ in} = \Theta_0\ . \tag{12}
$$

Some figures may be found in Klačka (1992b).

By inserting Eqs. (8) into Eq. (3), taking into account that $e_{\beta\ in} < 1$ (see Eq. (10)), one can easily obtain the secular changes of orbital elements:

$$
\langle\frac{da_\beta}{dt}\rangle = -\ \beta\ \frac{\mu}{c}\ \frac{2+3e_\beta^2}{a_\beta\ \left(1-e_\beta^2\right)^{3/2}}\ , \tag{13}
$$

$$
\langle\frac{de_\beta}{dt}\rangle = -\ \frac{5}{2}\ \beta\ \frac{\mu}{c}\ \frac{e_\beta}{a_\beta^2\ \left(1-e_\beta^2\right)^{1/2}}\ , \tag{14}
$$

$$\langle \frac{d\Theta_\beta}{dt} \rangle = \frac{\sqrt{\mu\,(1-\beta)}}{a_\beta^{3/2}} \ , \tag{15}$$

$$\langle \frac{di_\beta}{dt} \rangle = \langle \frac{d\Omega_\beta}{dt} \rangle = \langle \frac{d\omega_\beta}{dt} \rangle = 0 \ . \tag{16}$$

3.1. Advancement of perihelion, v^2/c^2-terms

Working to first order, as above, in v/c yields no secular change of ω (advancement of perihelion in Solar System). Thus, we have to include higher orders. Including now terms up to v^2/c^2, Eq. (1) yields

$$\frac{d\boldsymbol{v}}{dt} = \beta \ \frac{\mu}{r^2} \left\{ - \ \frac{1}{2} \left(\frac{\boldsymbol{v}}{c} \right)^2 \boldsymbol{S} + \left(\frac{\boldsymbol{v} \cdot \boldsymbol{S}}{c} \right) \frac{\boldsymbol{v}}{c} \right\} \ , \tag{17}$$

which gives

$$F_{\beta \ R} = \beta \frac{\mu}{r^2} \frac{v_R^2 - v_T^2}{2c^2} \ , \ F_{\beta \ T} = \beta \frac{\mu}{r^2} \frac{v_R v_T}{c^2} \ . \tag{18}$$

In reality, these components of perturbation acceleration represent only a part of the total perturbation acceleration, since general relativity theory produces other terms. The total advancement of pericenter is given (compare Balek and Klačka 2002) as

$$\langle \frac{d\omega_\beta}{dt} \rangle = \frac{3\mu^{3/2}}{c^2 a_\beta^{5/2} \left(1 - e_\beta^2 \right)} \frac{1 + \beta^2 \left(-13/8 + 7/e_\beta \right)/3}{\left(1 - \beta \right)^{1/2}} \ . \tag{19}$$

It can be easily verified that $< d\omega_\beta \ / \ dt >$ is i) an increasing function of β, ii) the perihelion circulates in a positive direction, iii) the rate of the advancement of perihelion is not bounded for $\beta \rightarrow 1$.

4. Electromagnetic radiation and general equation of motion

General equation of motion of a particle under the action of electromagnetic radiation can be written in the form (Klačka 2002c)

$$\frac{d \ p^\mu}{d \ \tau} = \sum_{j=1}^{3} \left(\frac{w_1^2 \ S \ A'}{c} \ Q'_j \ + \ Q'_{ej} \right) \ \left(b_j^\mu \ - \ u^\mu \ / \ c \right) \ , \tag{20}$$

where also nonradial components (in the proper frame of reference of the particle) of radiation pressure are considered and also thermal emission of dust particle is taken into account ($dm/d\tau = 0$).

Let us consider orbital evolution of real dust particle, under the action of gravitational and electromagnetic radiation fields of the central body (star). We can write for the Solar

System (Klačka 2002c)

$$\frac{d\,\boldsymbol{v}}{d\,t} = -\,\frac{G\,M_\odot}{r^2}\,\boldsymbol{e}_1 \;+\; \frac{G\,M_\odot}{r^2}\,\sum_{j=1}^{3}\,\beta_j\,\left[\left(1\;-\;2\,\frac{\boldsymbol{v}\cdot\boldsymbol{e}_1}{c}\;+\;\frac{\boldsymbol{v}\cdot\boldsymbol{e}_j}{c}\right)\,\boldsymbol{e}_j\;-\;\frac{\boldsymbol{v}}{c}\right]\;+\; \sum_{j=1}^{3}\,Q'_{ej}\,\left[\left(1\;+\;\frac{\boldsymbol{v}\cdot\boldsymbol{e}_j}{c}\right)\,\boldsymbol{e}_j\;-\;\frac{\boldsymbol{v}}{c}\right]\;, \tag{21}$$

where

$$\begin{aligned}
\boldsymbol{e}_j \;&=\; (1\;-\;\boldsymbol{v}\cdot\boldsymbol{e}'_j/c)\,\boldsymbol{e}'_j\;+\;\boldsymbol{v}/c\;,\quad j=1,2,3\;,\\
\beta_1 \;&=\; \frac{\pi\,R_\odot^2}{G\,M_\odot\,m\,c}\int_0^\infty B_\odot(\lambda)\,\{C'_{ext}(\lambda/w) - C'_{sca}(\lambda/w)\;g'_1(\lambda/w)\}\,d\lambda\;,\\
\beta_2 \;&=\; \frac{\pi\,R_\odot^2}{G\,M_\odot\,m\,c}\int_0^\infty B_\odot(\lambda)\,\{\,-\,C'_{sca}(\lambda/w)\;g'_2(\lambda/w)\}\,d\lambda\;,\\
\beta_3 \;&=\; \frac{\pi\,R_\odot^2}{G\,M_\odot\,m\,c}\int_0^\infty B_\odot(\lambda)\,\{\,-\,C'_{sca}(\lambda/w)\;g'_3(\lambda/w)\}\,d\lambda\;,\\
w \;&=\; 1\;-\;\boldsymbol{v}\cdot\boldsymbol{e}_1\;/\;c\;,
\end{aligned} \tag{22}$$

$R_\odot$ denotes the radius of the Sun and $B_\odot(\lambda)$ is the solar radiance at a wavelength of λ; G, $M_\odot$, and r are the gravitational constant, the mass of the Sun, and the distance of the particle from the center of the Sun, respectively. The asymmetry parameter vector $\boldsymbol{g'}$ is defined by $\boldsymbol{g'} = (1/C'_{sca})\int \boldsymbol{n'}(dC'_{sca}/d\Omega')d\Omega'$, where $\boldsymbol{n'}$ is a unit vector in the direction of scattering; $\boldsymbol{g'} = g'_1\;\boldsymbol{e}'_1 + g'_2\;\boldsymbol{e}'_2 + g'_3\;\boldsymbol{e}'_3$, $\boldsymbol{e}'_1 = (1\;+\;\boldsymbol{v}\cdot\boldsymbol{e}_1/c)\;\boldsymbol{e}_1 - \boldsymbol{v}/c$, $\boldsymbol{e}'_i\cdot\boldsymbol{e}'_j = \delta_{ij}$.

The first set of Eqs. (22) is important from the point of view that general equation of motion represented by Eqs. (20) and (21) has to be reduced to the special case treated by Einstein (1905).

We see that the motion of real, arbitrarily shaped, particle may by much more complicated than that corresponding to the P-R effect; P-R effect is a special case of Eq. (20). To the first order in v/c, Eq. (21) reduces to Eq. (2) for $\beta_2 = \beta_3 = Q'_{e1} = Q'_{e2} = Q'_{e3} = 0$; $\beta_1 \equiv \beta$, when comparing Eq. (2) and Eq. (21). Motion of a real particle depends on optical properties of the particle, e. g., shape, chemical composition of the particle, mass distribution within the particle (porosity).

5. P-R effect and secular changes of orbital elements – gravitation as a central acceleration

If one would like to inspire with calculation od orbital elements as it was presented in section 3, he should consider $-\,\mu\;(1\;-\;\beta_1)\;\boldsymbol{S}\;/\;r^2$ as a central acceleration determining osculating orbital elements even for general case represented by Eq. (21). However, the quantity β_1 depends on optical properties of the particle and, thus, also on particle's position with respect to the source of electromagnetic radiation.

It is not wise to use $- \mu \ (1 \ - \ \beta_1) \ \boldsymbol{S} \ / \ r^2$ as a central acceleration determining osculating orbital elements for general case represented by Eq. (21), since $\mu \ (1 \ - \ \beta_1)$ changes almost randomly during a motion. Thus, it is not wise to use $- \mu \ (1 \ - \ \beta) \ \boldsymbol{S} \ / \ r^2$ as a central acceleration determining osculating orbital elements for the P-R effect, if we want to compare the evolution of orbital elements for the P-R effect and the evolution of orbital elements for general case represented by Eq. (21). We need something which is not changing almost randomly. The wise quantity is μ. Thus, central acceleration $- \mu \ \boldsymbol{S} \ / \ r^2$ will be used as a central acceleration determining osculating orbital elements.

5.1. P-R effect and perturbation oquations of celestial mechanics

We use $- \mu \ \boldsymbol{S} \ / \ r^2$ as a central acceleration determining osculating orbital elements.

Taking into account Eq. (2), we take the action of electromagnetic radiation as a pertubation to the two-body problem. We can immediately write for components of perturbation acceleration:

$$F_R = \beta \ \frac{\mu}{r^2} \ - \ 2 \ \beta \ \frac{\mu}{r^2} \ \frac{v_R}{c} \ , \quad F_T = - \ \beta \ \frac{\mu}{r^2} \ \frac{v_T}{c} \ , \quad F_N = 0 \ , \tag{23}$$

where F_R, F_T and F_N are radial, transversal and normal components of perturbation acceleration, and two-body problem yields

$$\begin{aligned} v_R \ &= \ \sqrt{\frac{\mu}{p}} \ e \sin f \ , \\ v_T \ &= \ \sqrt{\frac{\mu}{p}} \ (1 + e \cos f) \ . \end{aligned} \tag{24}$$

Perturbation equations of celestial mechanics yield for osculating orbital elements (a – semi-major axis; e – eccentricity; i – inclination (of the orbital plane to the reference frame); Ω – longitude of the ascending node; ω – longitude of pericenter; Θ – $\Theta = \omega + f$ is the position angle of the particle on the orbit, when measured from the ascending node in the direction of the particle's motion):

$$\begin{aligned} \frac{da}{dt} \ &= \ \frac{2 \ a}{1 \ - \ e^2} \ \sqrt{\frac{p}{\mu}} \ \left\{ F_R \ e \ \sin f + F_T \ (1 \ + \ e \ \cos f) \right\} \ , \\ \frac{de}{dt} \ &= \ \sqrt{\frac{p}{\mu}} \ \left\{ F_R \ \sin f + F_T \left[\cos f \ + \ \frac{e + \cos f}{1 + e \cos f} \right] \right\} \ , \\ \frac{di}{dt} \ &= \ \frac{r}{\sqrt{\mu \ p}} \ F_N \ \cos \Theta \ , \\ \frac{d\Omega}{dt} \ &= \ \frac{r}{\sqrt{\mu \ p}} \ F_N \ \frac{\sin \Theta}{\sin i} \ , \end{aligned}$$

$$\frac{d\omega}{dt} = - \frac{1}{e} \sqrt{\frac{p}{\mu}} \left\{ F_R \cos f - F_T \frac{2 + e \cos f}{1 + e \cos f} \sin f \right\} - \frac{r}{\sqrt{\mu \ p}} \ F_N \ \frac{\sin \Theta}{\sin i} \ \cos i \ ,$$

$$\frac{d\Theta}{dt} = \frac{\sqrt{\mu \ p}}{r^2} - \frac{r}{\sqrt{\mu \ p}} \ F_N \ \frac{\sin \Theta}{\sin i} \ \cos i \ , \tag{25}$$

where $r = p / (1 + e \cos f)$.

Inserting Eqs. (23) – (24) into Eq. (25), one easily obtains

$$\begin{aligned}
\frac{da}{dt} &= \frac{2 \ \beta}{r^2} \sqrt{\frac{\mu \ a^3}{1 \ - \ e^2}} \ e \ \sin f \ - \ \beta \frac{\mu}{r^2} \ \frac{2 \ a}{c} \ \frac{1 + e^2 + 2e \cos f + e^2 \sin^2 f}{1 \ - \ e^2} \ , \\
\frac{de}{dt} &= \beta \ \frac{\sqrt{\mu \ p}}{r^2} \ \sin f \ - \ \beta \ \frac{\mu}{r^2} \frac{1}{c} \left(2e + e \sin^2 f + 2 \cos f \right) \ , \\
\frac{di}{dt} &= 0 \ , \\
\frac{d\Omega}{dt} &= 0 \ , \\
\frac{d\omega}{dt} &= - \ \beta \ \frac{\sqrt{\mu \ p}}{r^2} \ \frac{1}{e} \ \cos f \ - \ \beta \ \frac{\mu}{r^2} \ \frac{1}{c} \frac{1}{e} \ \left(2 - e \cos f \right) \ \sin f \ , \\
\frac{d\Theta}{dt} &= \frac{\sqrt{\mu \ p}}{r^2} \ .
\end{aligned} \tag{26}$$

It is worth mentioning that $da/dt < 0$ for any time t does not hold (perturbation corresponds to complete nongravitational acceleration, and, thus, Eqs. (8) do not hold).

The set of differential equations Eqs. (26) has to be complemented with initial conditions. If the subscript 0 denotes orbital elements of the parent body (e. g., comet), then particle ejected with zero relative velocity will have initial osculating orbital elements identical with those of the parent body:

$$\begin{aligned}
a_{in} &= a_0 \ , \ \ e_{in} = e_0 \ , \ \ i_{in} = i_0 \ , \\
\Omega_{in} &= \Omega_0 \ , \ \ \omega_{in} = \omega_0 \ , \ \ \Theta_{in} = \Theta_0 \ .
\end{aligned} \tag{27}$$

The important fact is that Eqs. (26) contain also terms not proportional to v/c ($\ll 1$). These important terms protect us to use procedure analogous to that represented by Eq. (3). While dispersion of osculating orbital elements is very small during a time interval T for the case when $\mu \ (1 - \beta)$ is used in central acceleration, the dispersion of osculating orbital elements may be large during the same time interval for the case when μ is used in central acceleration. Thus, any formal averaging of Eqs. (26) leading to equations analogous to Eqs. (13)-(16) is not correct. This fact was discussed in Klačka (1992b), see also Figs. 1 and 2 in Klačka (1994).

5.2. Expressions for secular changes of semi-major axis and eccentricity

We have already explained that it is not allowed to make a simple time averaging analogous to that described by Eq. (3), when $- \ \mu \ \boldsymbol{S} \ / \ r^2$ is used as a central acceleration determining osculating orbital elements. However, it is of interest if we have to numerically solve Newtonian vectorial equation of motion (Eq. (2)) and make numerical time averaging (over a time interval between passages through two following pericenters), or some analytical simplifications can be done – something analogous to Eqs. (13)-(14).

Correct answer is: yes, we can make some analytical simplifications when we want to obtain secular changes of semi-major axis and eccentricity even when $- \mu \, \boldsymbol{S} \, / \, r^2$ is used as a central acceleration determining osculating orbital elements. We will derive the correct equations in the following two subsections.

5.2.1. Radial forces and orbital elements

We will proceed according to Klačka (1994), in this subsection.

Let us consider a gravitational system of two bodies

$$\dot{\boldsymbol{v}} = - \, \frac{\mu}{r^2} \, \boldsymbol{e}_R \ , \tag{28}$$

where $\boldsymbol{e}_R \equiv \boldsymbol{e}_1 \equiv \boldsymbol{S}$. Let perturbation acceleration exists in the form

$$\boldsymbol{a} = \beta \, \frac{\mu}{r^2} \, \boldsymbol{e}_R \ , \tag{29}$$

$0 \le \beta < 1$. Thus, the final equation of motion is

$$\dot{\boldsymbol{v}} = - \, \frac{\mu \ (1 \ - \ \beta)}{r^2} \, \boldsymbol{e}_R \ . \tag{30}$$

Eq. (30) yields as a solution the well-known Keplerian motion and the orbit is given by

$$r = \frac{p_c}{1 \ + \ e_c \ \cos \left(\Theta \ - \ \omega_c \right)} \ , \tag{31}$$

where

$$p_c = a_c \ \left(1 \ - \ e_c^2 \right) \ . \tag{32}$$

The subscript "c" denotes that orbital elements are constants of motion. If we write

$$\boldsymbol{v} = v_R \, \boldsymbol{e}_R \ + \ v_T \, \boldsymbol{e}_T \ , \tag{33}$$

where $\boldsymbol{e}_T$ is a unit vector transverse to the radial vector $\boldsymbol{e}_R$ in the plane of the trajectory (positive in the direction of motion), we have

$$v_R = \sqrt{\mu \ (1 \ - \ \beta) \ p_c^{-1}} \ e_c \ \sin \left(\Theta \ - \ \omega_c \right) \ , \tag{34}$$

$$v_T = \sqrt{\mu \ (1 \ - \ \beta) \ p_c^{-1}} \ \left[1 \ + \ e_c \ \cos \left(\Theta \ - \ \omega_c \right) \right] \ . \tag{35}$$

In principle, we may consider also a new set of orbital elements, which are defined by the central gravitational acceleration. Equations (31), (32), (34) and (35) are then of the form

$$r = \frac{p}{1 \, + \, e \, \cos \left(\Theta \, - \, \omega \right)} \ , \tag{36}$$

$$p = a \, \left(1 \, - \, e^2 \right) \ , \tag{37}$$

$$v_R = \sqrt{\mu \, p^{-1}} \, e \, \sin \left(\Theta \, - \, \omega \right) \ , \tag{38}$$

$$v_T = \sqrt{\mu \, p^{-1}} \, \left[1 \, + \, e \, \cos \left(\Theta \, - \, \omega \right) \right] \ ; \tag{39}$$

the fact that Θ is unchanged in both sets of orbital elements is used.

Position vector and velocity vector define a state of the body at any time. Equations (31) and (36) yield then

$$\frac{p_c}{1 \, + \, e_c \, \cos \left(\Theta \, - \, \omega_c \right)} = \frac{p}{1 \, + \, e \, \cos \left(\Theta \, - \, \omega \right)} \ . \tag{40}$$

Analogously, the other two pairs of equations (Eqs. (34) and (38), and, Eqs. (35) and (39)) give

$$\sqrt{\left(1 \, - \, \beta \right) \, p_c^{-1}} \, e_c \, \sin \left(\Theta \, - \, \omega_c \right) = \sqrt{p^{-1}} \, e \, \sin \left(\Theta \, - \, \omega \right) \ , \tag{41}$$

$$\sqrt{\left(1 \, - \, \beta \right) \, p_c^{-1}} \, \left[1 \, + \, e_c \, \cos \left(\Theta \, - \, \omega_c \right) \right] = \sqrt{p^{-1}} \, \left[1 \, + \, e \, \cos \left(\Theta \, - \, \omega \right) \right] \ . \tag{42}$$

One can easily obtain, using Eqs. (40) and (42),

$$p_c \, \left(1 \, - \, \beta \right) = p \ , \tag{43}$$

and Eqs. (41)-(42) yield then

$$\left(1 \, - \, \beta \right) \, e_c \, \sin \left(\Theta \, - \, \omega_c \right) = e \, \sin \left(\Theta \, - \, \omega \right) \ , \tag{44}$$

$$\left(1 \, - \, \beta \right) \, \left[1 \, + \, e_c \, \cos \left(\Theta \, - \, \omega_c \right) \right] = 1 \, + \, e \, \cos \left(\Theta \, - \, \omega \right) \ . \tag{45}$$

Eq. (45) can be written in the form

$$\left(1 \, - \, \beta \right) \, e_c \, \cos \left(\Theta \, - \, \omega_c \right) \, - \, \beta = e \, \cos \left(\Theta \, - \, \omega \right) \ . \tag{46}$$

Eqs. (44) and (46) yield

$$e^2 = \left(1 \, - \, \beta \right)^2 \, e_c^2 \, + \, \beta^2 \, - \, 2 \, \beta \, \left(1 \, - \, \beta \right) \, e_c \, \cos \left(\Theta \, - \, \omega_c \right) \ . \tag{47}$$

Equation for ω is given by Eqs. (44) and (46), using also Eq. (47). Finally, Eqs. (32), (37), (43) and (47) yield

$$a = a_c \left\{ 1 \ + \ \beta \ \frac{1 \ + \ e_c^2 \ + \ 2 \ e_c \ \cos(\Theta \ - \ \omega_c)}{1 \ - \ e_c^2} \right\} \quad . \tag{48}$$

Eqs. (47)-(48) show that orbital osculating elments a and e change in time, during an orbital revolution – the larger β, the larger change of a and e.

The osculating orbital elements e and a obtain values between their maxima and minima, which can be easily found from Eqs. (47)-(48). One can easily verify that these relations hold:

$$e_{min} = | \, (1 \ - \ \beta) \ e_c \ - \ \beta | \le e \le (1 \ - \ \beta) \ e_c \ + \ \beta = e_{max} \ , \tag{49}$$

$$\frac{a_{min}}{a_c} = \frac{1 \ - \ e_c}{1 \ - \ e_c \ + \ \beta \ (1 \ + \ e_c)} \le \frac{a}{a_c} \le \frac{1 \ + \ e_c}{1 \ + \ e_c \ + \ \beta \ (1 \ - \ e_c)} = \frac{a_{max}}{a_c} \ . \tag{50}$$

5.2.2. Mean values of semi-major axis and eccentricity

Eqs. (49) and (50) represent interval of values for eccentricity and semi-major axis, when $- \ \mu \ \boldsymbol{S} \ / \ r^2$ is used as a central acceleration determining osculating orbital elements. However, we can make time averaging during a period T, which was decribed by Eq. (3):

$$\begin{aligned} & \langle e \rangle \ = \ \frac{1}{T} \int_0^T e(t) \ dt \ , \quad \langle a \rangle = \frac{1}{T} \int_0^T a(t) \ dt \ , \\ & r^2 \ \frac{df_c}{dt} \ = \ \sqrt{\mu \ (1 \ - \ \beta) \ p_c} \ , \quad \frac{a_c^3}{T^2} = \frac{\mu \ (1 \ - \ \beta)}{4 \ \pi^2} \ , \quad r = \frac{p_c}{1 \ + \ e_c \ \cos f_c} \ . \end{aligned} \tag{51}$$

Eqs. (51) yield

$$\begin{aligned} \langle e \rangle \ &= \ \left(1 \ - \ e_c^2\right)^{3/2} \ \frac{1}{2 \ \pi} \int_0^{2\pi} \frac{e \ (f_c)}{\left(1 \ + \ e_c \ \cos f_c\right)^2} \ df_c \ , \\ \langle a \rangle \ &= \ \left(1 \ - \ e_c^2\right)^{3/2} \ \frac{1}{2 \ \pi} \int_0^{2\pi} \frac{a \ (f_c)}{\left(1 \ + \ e_c \ \cos f_c\right)^2} \ df_c \ . \end{aligned} \tag{52}$$

Using Eqs. (47) and (48), we finally obtain

$$\begin{aligned} \langle e \rangle \ &= \ \left(1 - e_c^2\right)^{3/2} \ \frac{1}{2\pi} \int_0^{2\pi} \frac{\sqrt{(1-\beta)^2 \, e_c^2 + \beta^2 - 2\beta \, (1-\beta) \, e_c \cos f_c}}{\left(1 + e_c \cos f_c\right)^2} \ df_c \ , \\ \langle a \rangle \ &= \ a_c \left(1 - e_c^2\right)^{3/2} \ \frac{1}{2\pi} \int_0^{2\pi} \frac{\left[1 + \beta \left(1 + e_c^2 + 2 e_c \cos f_c\right) / \left(1 - e_c^2\right)\right]^{-1}}{\left(1 + e_c \cos f_c\right)^2} \ df_c \ . \end{aligned} \tag{53}$$

The following properties can be verified:

i) $\langle e \rangle \ge \beta$, $\langle e \rangle = \beta$ for $e_c = 0$; ii) $\langle e \rangle / e_c \ge 1$, $\langle e \rangle = e_c$ for $\beta = 0$;

iii) $\partial \langle e \rangle / \partial e_c > 0$; iv) $\partial \langle e \rangle / \partial \beta > 0$;

v) $\langle a \rangle \ge a_c / (1 + \beta)$, $\langle a \rangle = a_c / (1 + \beta)$ for $e_c = 0$;

vi) $\langle a \rangle / a_c \le 1$, $\langle a \rangle = a_c$ for $\beta = 0$;

vii) $\partial \langle a \rangle / \partial e_c > 0$; viii) $\partial \langle a \rangle / \partial \beta < 0$; ix) $\partial \langle a \rangle / \partial a_c > 0$.

5.2.3. Secular changes of semi-major axis and eccentricity

Summarizing our results, it is possible to calculate secular evolution of eccentricity and semi-major axis, according to the following prescription.

At first, initial conditions for a_β and e_β are calculated:

$$
\begin{aligned}
\left(a_\beta\right)_{in} &= a_0 \left(1-\beta\right) \left(1 - 2\beta \frac{1+e_0 \cos f_0}{1 - e_0^2}\right)^{-1} , \\
\left(e_\beta\right)_{in} &= \sqrt{1 - \frac{1-e_0^2 - 2\beta\left(1+e_0\cos f_0\right)}{\left(1-\beta\right)^2}} ,
\end{aligned}
\tag{54}
$$

supposing that particle was ejected with zero ejection velocity from a parent body – quantities with subscript 0 belongs to the parent body's trajectory.

As the second step, the set of the following differential equations must be solved for the above presented initial conditions:

$$
\begin{aligned}
\frac{da_\beta}{dt} &= -\,\beta\,\frac{\mu}{c}\,\frac{2+3e_\beta^2}{a_\beta\,\left(1-e_\beta^2\right)^{3/2}} , \\
\frac{de_\beta}{dt} &= -\,\frac{5}{2}\,\beta\,\frac{\mu}{c}\,\frac{e_\beta}{a_\beta^2\,\left(1-e_\beta^2\right)^{1/2}} .
\end{aligned}
\tag{55}
$$

Finally, semi-major axis and eccentricity are calculated from:

$$
\begin{aligned}
a &= a_\beta \left(1-e_\beta^2\right)^{3/2}\,\frac{1}{2\pi}\,\int_0^{2\pi} \frac{\left[1+\beta\left(1+e_\beta^2+2e_\beta\cos x\right)/\left(1-e_\beta^2\right)\right]^{-1}}{\left(1+e_\beta\cos x\right)^2}\,dx , \\
e &= \left(1-e_\beta^2\right)^{3/2}\,\frac{1}{2\pi}\,\int_0^{2\pi} \frac{\sqrt{\left(1-\beta\right)^2 e_\beta^2 + \beta^2 - 2\beta\left(1-\beta\right)e_\beta\cos x}}{\left(1+e_\beta\cos x\right)^2}\,dx .
\end{aligned}
\tag{56}
$$

The set of equations represented by Eqs. (54)-(56) fully corresponds to detailed numerical calculations of vectorial equation of motion, if we are interested in secular evolution of eccentricity and semi-major axis (supposing $(e_\beta)_{in} < 1$ and e_β does not correspond to pseudo-circular orbit) for the case when central acceleration is defined by gravity alone.

It is worth mentioning that instantaneous time derivatives of semi-major axis and eccentricity may be both positive and negative (see Eqs. (26)), while secular evolution yields that semi-major axis and eccentricity are decreasing functions of time.

6. Discussion

When the orbits are near circular (pseudo-circular) – central acceleration contains radiation pressure term – the orbit can not reduce in semi-major axis without increasing in eccentricity. Both types of orbital elements, defined by central accelerations, have been used in detail numerical calculations in papers by Klačka and Kaufmannová (1992, 1993). Due to the property $\langle e \rangle \geq \beta$ (see section 5.2.2), one must be aware that even value $\langle e \rangle > \beta$ may correspond to pseudo-circular orbit. The results for pseudo-circular orbits were analytically confirmed by Breiter and Jackson (1998). Although the analytical approach reproduces known results without detail numerical calculations, it paralelly produces nonphysical results which may not be distinguishable from the correct results. The nonphysical analytical results are caused by usage of P-R effect in the first order in v/c – very special analytical solutions will diminish when higher orders in v/c are used. The nonphysical analytical results have been discussed in more detail by Klačka (2000a, 2001a).

7. Conclusion

We have discussed orbital evolution for the P-R effect. General equation of motion for interaction between particle and incident electromagnetic radiation shows that radiation cannot be considered as a part of central acceleration. The central acceleration has to contain only gravity of the central body (star). In order to compare secular changes of semi-major axis and eccentricity for real particle and the P-R effect, the paper presents secular changes of these orbital elements for the P-R effect (Eqs. (54)-(56)).

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References

Balek V., Klačka J., 2002, Perihelion motion due to the Poynting-Robertson effect. http://xxx.lanl.gov/abs/astro-ph/0207496

Breiter S., Jackson A. A., 1998, Unified analytical solutions to two-body problems with drag. *Mon. Not. R. Astron. Soc.* **299**, 237-243.

Einstein A., 1905, Zur Elektrodynamik der bewegter Kőrper. *Annalen der Physik* **17**, 891-920.

Gajdošík M., Klačka J., 1999, Cometary dust trails and ejection velocities. In: Evolution and Source Regions of Asteroids and Comets, Proc. IAU Coll. 173, J. Svoreň, E. M. Pittich, and H. Rickman (eds.), Astron. Inst. Slovak Acad. Sci., Tatranská Lomnica, pp. 239-242.

Harwit M., 1963, Origins of the Zodiacal dust cloud. *J. Geophys. Res.* **86**, 2171-2180.

Klačka J., 1992a, Poynting-Robertson effect. I. Equation of motion. *Earth, Moon, and Planets* **59**, 41-59.

Klačka J., 1992b, Poynting-Robertson effect. II. Perturbation equations. *Earth, Moon, and Planets* **59**, 211-218.

Klačka J., 1993a, Aberration of light and the Poynting-Robertson effect. *Earth, Moon, and Planets* **62**, 239-244.

Klačka J., 1993b, Misunderstanding of the Poynting-Robertson effect. *Earth, Moon, and Planets* **63**, 255-258.

Klačka J., 1994, Radial forces and orbital elements. In: *Dynamics and Astrometry of Natural and Artificial Celestial Bodies*, K. Kurzynska, F. Barlier, P. K. Seidelmann and I. Wytrzyszczak (eds.), Astronomical Observatory of A. Mickiewicz University, Poznan, pp. 181-185.

Klačka J., 2000a, On the Poynting-Robertson effect and analytical solutions. http://xxx.lanl.gov/abs/astro-ph/0004181

Klačka J. 2000b. On physical interpretation of the Poynting-Robertson effect. http://xxx.lanl.gov/abs/astro-ph/0006426

Klačka J., 2000c, Electromagnetic radiation and motion of real particle. http://xxx.lanl.gov/abs/astro-ph/0008510

Klačka J., 2000d, Aberration of light and motion of real particle. http://xxx.lanl.gov/abs/astro-ph/0009108

Klačka J., 2000e, Solar radiation and asteroidal motion. http://xxx.lanl.gov/abs/astro-ph/0009109

Klačka J., 2001a, On the Poynting-Robertson effect and analytical solutions. In: Dynamics of Natural and Artificial Celestial Bodies, H. Pretka-Ziomek, E. Wnuk, P. K. Seidelmann and D. Richardson (eds.), Kluwer Academic Publishers, Dordrecht, pp. 215-217.

Klačka J., 2001b, Motion of electrically neutral particle in the field of electromagnetic radiation. *Meteor Reports* **22**, 21-28.

Klačka J., 2002a, Covariant equation of motion for a particle in an electromagnetic field. In: *Optics of Cosmic Dust*, G. Videen and M. Kocifaj (eds.), Kluwer Academic Publishers, Dordrecht/Boston/London, pp. 301-312.

Klačka J., 2002b, Orbital elements for motion of real particle under the action of electromagnetic radiation. http://xxx.lanl.gov/abs/astro-ph/0201201

Klačka J., 2002c, Radiation and dynamics of dust particle. http://xxx.lanl.gov/abs/astro-ph/0209461 (submitted to Celestial Mechanics and Dynamical Astronomy)

Klačka J., Kaufmannová J., 1992, Poynting-Robertson effect: 'circular' orbit. *Earth, Moon, and Planets* **59**, 97-102.

Klačka J., Kaufmannová J., 1993, Poynting-Robertson effect and small eccentric orbits. *Earth, Moon, and Planets* **63**, 271-274.

Klačka J., Kocifaj M., 2001, Motion of nonspherical dust particle under the action of electromagnetic radiation. *J. Quant. Spectrosc. Radiat. Transfer* **70/4-6**, 595-610.

Robertson H. P., 1937, The dynamical effects of radiation in the Solar System. *Mon. Not. R. Astron. Soc.* **97**, 423-438.

Wyatt S. P., Whipple F. L., 1950, The Poynting-Robertson effect on meteor orbits. *Astrophys. J.* **111**, 558-565.

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